MILNOR INVARIANTS AND EDGE-HOMOTOPY CLASSIFICATION OF CLOVER LINKS

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Abstract. Given a clover link, we construct a bottom tangle by using a disk/band surface of the clover link. Since the Milnor number is already defined for a bottom tangle, we define the Milnor number for the clover link to be the Milnor number for the bottom tangle and show that for a clover link, if Milnor numbers of length $\leq k$ vanish, then Milnor numbers of length $\leq 2k+1$ are well-defined. Moreover we prove that two clover links whose Milnor numbers of length $\leq k$ vanish are equivalent up to edge-homotopy and C_{2k+1}-equivalence if and only if those Milnor numbers of length $\leq 2k+1$ are equal. In particular, we give an edge-homotopy classification of 3-clover links by their Milnor numbers of length ≤ 3.

1. Introduction

In 1954, J. Milnor [11] introduced a concept of *link-homotopy* which is a weaker equivalence relation than link type, where link-homotopy is an equivalence relation generated by crossing changes on the same component. And he [11], [12] defined *Milnor $\overline{\mu}$-invariants* which are given as follows. (See Subsection 2.3 for detail definitions.) Let L be an oriented ordered n-component link in S^3. The *Milnor number* $\mu_L(I)$ is an integer determined by a finite sequence I of numbers in $\{1,2,\ldots,n\}$. Let $\Delta_L(I)$ be the greatest common devisor of $\mu_L(J)$'s, where J is obtained from proper subsequence of I by permuting cyclicly. The Milnor $\overline{\mu}$-invariant $\overline{\mu}_L(I)$ is the residue class of $\mu_L(I)$ modulo $\Delta_L(I)$. The length of I is called the *length* of $\overline{\mu}_L(I)$ and denoted by $|I|$.

Milnor [11] gave a link-homotopy classification for 2- or 3-component links by Milnor $\overline{\mu}$-invariants. In 1988, J. P. Levine [9] gave a link-homotopy classification for 4-component links. In 1990, N. Habegger and X. S. Lin [3] gave an algorithm which determines if two links with arbitrarily many components are link-homotopic. In [3], they defined Milnor numbers for string links which are similarly defined as links and proved that Milnor numbers are invariants for string links. Moreover they showed that Milnor numbers give a link-homotopy classification for string links with arbitrarily many components. We remark that Milnor numbers are complete link-homotopy invariants for string links, but Milnor $\overline{\mu}$-invariants are not strong enough to classify for links up to link-homotopy.

An embedded graph in the 3-sphere S^3 is called a *spatial graph*. Let C_n be a graph consisting of n loops, each loop connected to a vertex by an edge. We call a spatial graph of C_n an *n-clover link* in S^3 [8]. Given an n-clover link c, we construct an n-component bottom tangle γ_{F_c} by using a *disk/band surface* F_c of c. In [8], Levine defined the Milnor number for a bottom tangle. Therefore we define the *Milnor number μ_c for an n-clover link c* to be the Milnor number $\mu_{\gamma_{F_c}}$. In [8], a bottom tangle is called a string link. (The name 'bottom tangle' follows K. Habiro [5].) In this paper, we mean that a string link is one defined in [3].

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There is a one-to-one correspondence between the sets of string links and bottom tangles. (See subsection 2.1.) This correspondence naturally induces the one-to-one correspondence between the Milnor number for the bottom tangles and the Milnor number for the string links.

We remark that there are infinitely many choices of γ_{F_c} for c, and hence that, in general, μ_c is not an invariant for c. But under a certain condition, μ_c is well-defined as follows.

Theorem (Theorem 2.2). Let c be an n-clover link and l_c a link which is the disjoint union of loops of c. If $\overline{\mu}_{l_c}(J) = 0$ for any sequence J with $|J| \leq k$, then $\mu_c(I)$ is well-defined for any sequence I with $|I| \leq 2k+1$.

In 1988, Levine [8] already defined Milnor numbers for *flat vertex* clover links and proved the same result as the theorem above, where a flat vertex spatial graph is a spatial graph Γ, for each vertex v of Γ, there exist a neighborhood B_v of v and a small flat plane P_v such that $\Gamma \cap B_v \subset P_v$ [15]. In this paper, we do not assume that clover links are flat vertex ones. And we consider that two clover links are *equivalent* if they are ambient isotopic.

By using Milnor numbers for clover links, we have the following results for an edge-homotopy classification of clover links, where edge-homotopy is an equivalence relation generated by crossing changes on the same spatial edge. This equivalence relation was introduced by K. Taniyama [14] as a generalization of link-homotopy.

Theorem (Theorem 3.3). Let c, c' be two n-clover links and $l_c, l_{c'}$ links which are disjoint unions of loops of c, c' respectively. Suppose that $\overline{\mu}_{l_c}(J) = \overline{\mu}_{l_{c'}}(J) = 0$ for any sequence J with $|J| \leq k$. Then c and c' are (edge-homotopy+C_{2k+1})-equivalence if and only if $\mu_c(I) = \mu_{c'}(I)$ for any non-repeated sequence I with $|I| \leq 2k+1$, where (edge-homotopy+C_{2k+1})-equivalence is an equivalence relation obtained by combining edge-homotopy and C_{2k+1}-equivalence which is defined by Habiro [4].

We also have the following proposition.

Proposition (Corollary 3.4). Let c, c' be two n-clover links and $l_c, l_{c'}$ links which are disjoint unions of loops of c, c' respectively. Suppose that $\overline{\mu}_{l_c}(J) = \overline{\mu}_{l_{c'}}(J) = 0$ for any sequence J with $|J| \leq n/2$. Then c and c' are edge-homotopic if and only if $\mu_c(I) = \mu_{c'}(I)$ for any non-repeated sequence I with $|I| \leq n$.

It is the definition that the Milnor $\overline{\mu}$-invariant of length 1 is zero. If $n = 3$, then the proposition above holds without the condition.

Corollary (Corollary 3.5). Two 3-clover links c and c' are edge-homotopic if and only if $\mu_c(I) = \mu_{c'}(I)$ for any non-repeated sequence I with $|I| \leq 3$.

Remark 1.1. Let c be a clover link and l_c a link which is the disjoint union of loops of c. By the definition of μ_c in subsection 2.3, we note that $\mu_c(I) = 0$ for any non-repeated sequence I if and only if $\mu_{l_c}(I) = 0$ for I. And we also note that all Milnor numbers for a trivial clover link vanish, where a clover link is *trivial* if there is an embedded plane in S^3 which contains the clover link. It is shown by Milnor [11] that a link is link-homotopic to a trivial link if and only if the Milnor $\overline{\mu}$-invariant vanishes for any non-repeated sequence. Hence, by the proposition above, we have that c is edge-homotopic to a trivial clover link if and only if l_c is link-homotopic to a trivial link.

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2. Milnor numbers for clover links

In this section we will define Milnor numbers for clover links.

2.1. **Tangles.** An n-component *tangle* is a properly embedded disjoint union of n arcs in the 3-cube $[0,1]^3$. An n-component tangle $sl = sl_1 \cup sl_2 \cup \cdots \cup sl_n$ is an n-component *string link* if for each i $(= 1, 2, \ldots, n)$, the boundary $\partial sl_i = \{(\frac{2i-1}{2n+1}, \frac{1}{2}, 0), (\frac{2i-1}{2n+1}, \frac{1}{2}, 1)\} \subset \partial[0,1]^3$. In particular, sl is *trivial* if for each i $(= 1, 2, \ldots, n)$, $sl_i = \{(\frac{2i-1}{2n+1}, \frac{1}{2})\} \times [0,1]$ in $[0,1]^3$.

Product of n-component string links is defined as follows. Let $sl = sl_1 \cup sl_2 \cup \cdots \cup sl_n$ and $sl' = sl'_1 \cup sl'_2 \cup \cdots \cup sl'_n$ be two string links in $[0,1]^3$. Then the *product* $sl * sl' = (sl_1 * sl'_1) \cup (sl_2 * sl'_2) \cup \cdots \cup (sl_n * sl'_n)$ of sl and sl' is a string link in $[0,1]^3$ defined by

$$sl_i * sl'_i = h_0(sl_i) \cup h_1(sl'_i)$$

for $i = 1, 2, \ldots, n$, where $h_0, h_1 : ([0,1] \times [0,1]) \times [0,1] \to ([0,1] \times [0,1]) \times [0,1]$ are embeddings defined by

$$h_0(x,t) = (x, \tfrac{1}{2}t) \text{ and } h_1(x,t) = (x, \tfrac{1}{2} + \tfrac{1}{2}t)$$

for $x \in ([0,1] \times [0,1])$ and $t \in [0,1]$, see Figure 2.1.

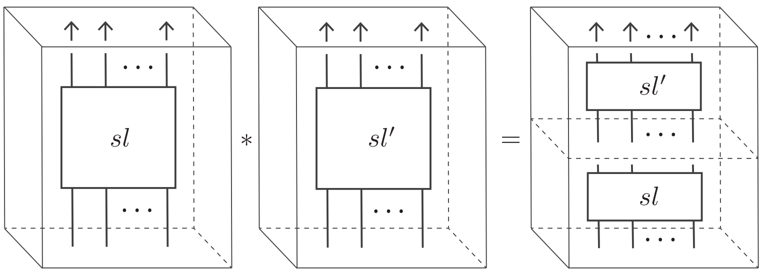
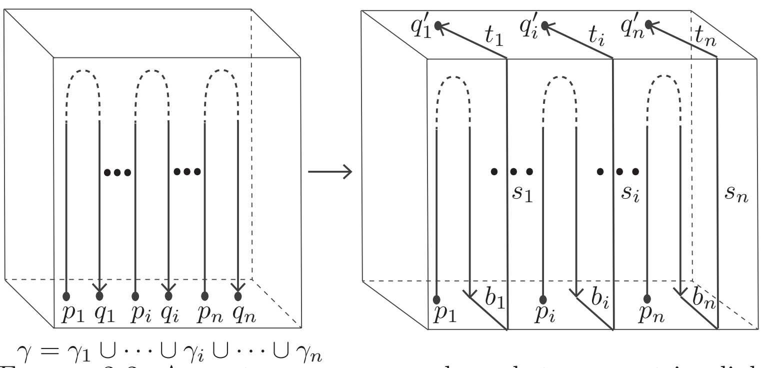


Figure 2.1. A product of two string links sl and sl'



$\gamma = \gamma_1 \cup \cdots \cup \gamma_i \cup \cdots \cup \gamma_n$

Figure 2.2. A one-to-one correspondence between a string link and a bottom tangle

An n-component *bottom tangle* $\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ defined by Levine [8] is a tangle with $\partial\gamma_i = \{(\frac{2i-1}{2n+1}, \frac{1}{2}, 0), (\frac{2i}{2n+1}, \frac{1}{2}, 0)\} \subset \partial[0,1]^3$ for each i $(= 1, 2, \ldots, n)$.

We explain that there is a one-to-one correspondence between the sets of string links and bottom tangles in the following. We first describe a construction of obtaining a string link from a bottom tangle $\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$. For each i $(= 1, 2, \ldots, n)$, let p_i and q_i be the end points $(\frac{2i-1}{2n+1}, \frac{1}{2}, 0)$ and $(\frac{2i}{2n+1}, \frac{1}{2}, 0)$ of γ_i respectively, and let q'_i be the point $(\frac{2i-1}{2n+1}, \frac{1}{2}, 1)$. Let b_i be a line segment between

q_i and $(\frac{2i}{2n+1}, 0, 0)$, s_i a line segment between $(\frac{2i}{2n+1}, 0, 0)$ and $(\frac{2i}{2n+1}, 0, 1)$ and t_i a line segment between $(\frac{2i}{2n+1}, 0, 1)$ and q'_i, see Figure 2.2. By pushing the strands $\gamma_i \cup b_i \cup s_i \cup t_i$ into the interior of $[0,1]^3$ with fixing the end points p_i and q'_i, we have a string link.

Conversely let $sl = sl_1 \cup sl_2 \cup \cdots \cup sl_n$ be a string link with $\partial sl_i = \{p_i, q'_i\}$ for each i $(= 1, 2, \ldots, n)$. By pushing the strands $sl_i \cup t_i \cup s_i \cup b_i$ into the interior of $[0,1]^3$ with fixing the end points p_i and q_i, we have a bottom tangle.

2.2. **A bottom tangle obtained from a disk/band surface of a clover link.** Let C_n be a graph consisting of n oriented loops $e_1, e_2, \ldots, e_n$, each loop e_i connected to a vertex v by an edge f_i $(i = 1, 2, \ldots, n)$, see Figure 2.3. An *n-clover link* in S^3 is a spatial graph of C_n [8]. The each part of a clover link corresponding to e_i, f_i and v of C_n are called the *leaf*, *stem* and *root*, denoted by the same notations respectively.

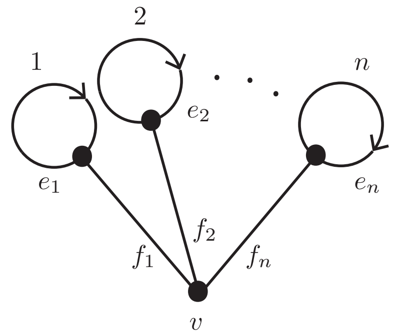


FIGURE 2.3. The graph C_n

L. Kauffman, J. Simon, K. Wolcott and P. Zhao [7] defined disk/band surfaces for spatial graphs. For a spatial graph Γ, a *disk/band surface* F_Γ of Γ is a compact, oriented surface in S^3 such that Γ is a deformation retract of F_Γ contained in the interior of F_Γ. Note that any disk/band surface of a spatial graph is ambient isotopic to a surface constructed by putting a disk at each vertex of the spatial graph, connecting the disks with bands along the spatial edges. We remark that for a spatial graph, there are infinitely many disk/band surfaces up to ambient isotopy.

Given an n-clover link, we construct an n-component bottom tangle using a disk/band surface of the clover link as follows.

(1) For an n-clover link c, let F_c be a disk/band surface of c and let D be a disk which contains the root. From now on, we may assume that the intersection $D \cap \bigcup_{i=1}^{n} f_i$ and orientations of the disks are as illustrated in Figure 2.4.
(2) Let $N(D)$ be the regular neighborhood of D.
(3) Since $S^3 \setminus N(D)$ is homeomorphic to the 3-ball, $F_c \setminus N(D)$ can be seen as a disjoint union of surfaces in the 3-ball. Hence $\partial F_c \setminus N(D)$ is a disjoint union of n-arcs and n-circles $\bigcup_{i=1}^{n} S^1_i$ in the 3-ball.
(4) Since the 3-ball is homeomorphic to $[0,1]^3$, we obtain an oriented ordered n-component bottom tangle γ_{F_c} from $(\partial F_c \setminus N(D)) \setminus \bigcup_{i=1}^{n} S^1_i$ as illustrated

in (3) and (4) of Figure 2.5. We call γ_{F_c} an *n-component bottom tangle obtained from* F_c.

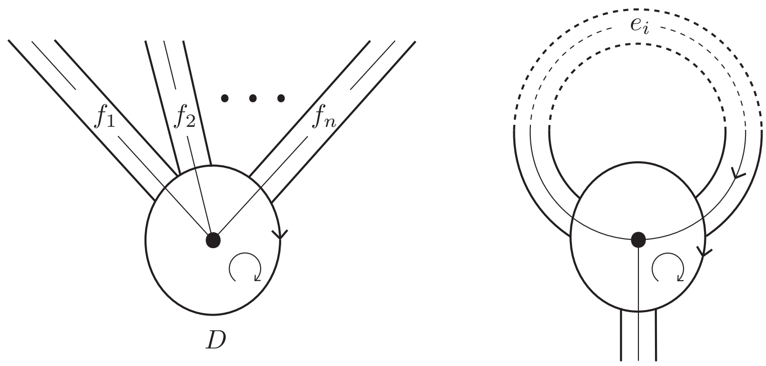


FIGURE 2.4.

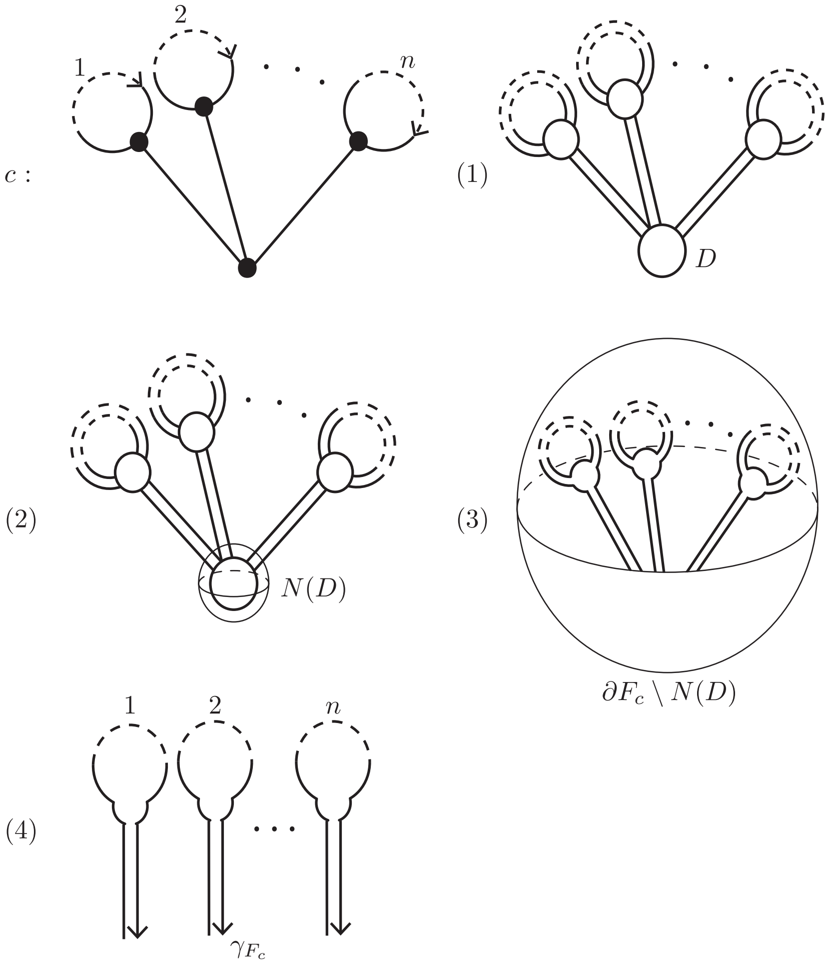


FIGURE 2.5. A method for obtaining a bottom tangle from a disk-band surface of a clover link

2.3. **Milnor invariants.** Let us briefly recall from [8] the definition of the Milnor number for a bottom tangle. Let $\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ be an oriented ordered n-component bottom tangle in $[0,1]^3$. Let G be the fundamental group of $[0,1]^3 \setminus \gamma$ with a base point $p = (\frac{1}{2}, 0, 0)$ and G_q the qth lower central subgroup of G, namely $G_1 = G$, G_q is the subgroup generated by $\{a^{-1}b^{-1}ab \mid a \in G, b \in G_{q-1}\}$. Then the quotient group G/G_q is generated by $\alpha_1, \alpha_2, \ldots, \alpha_n$ ([1], [13]), where α_i is the ith meridian of γ which is represented by the composite path $t_i m_i {t_i}^{-1}$ in the (x,y)-plane, m_i is a small counterclockwise circle about the point p_i and t_i is a straight line from p_i to m_i, see Figure 2.6. Then the ith longitude λ_i of γ is represented by $\alpha_1, \alpha_2, \ldots, \alpha_n$ modulo G_q, where λ_i is represented by the composite path $t_i l_i {t'_i}^{-1}$, t'_i is a straight line from p to the boundary of a small neighborhood of q_i and l_i is a path on the boundary of a small regular neighborhood of γ_i. We assume that λ_i is trivial in G/G_2. See Figure 2.7.

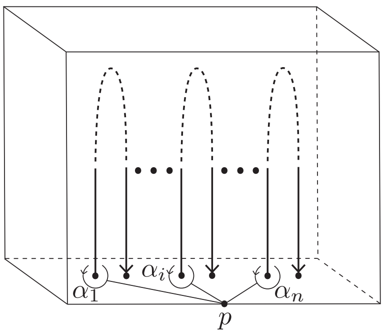


Figure 2.6. meridians

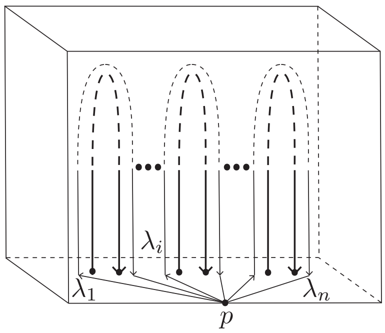


Figure 2.7. longitudes

We consider the Magnus expansion $E(\lambda_j)$ of λ_j. The Magnus expansion E is a homomorphism from a free group $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ to the formal power series ring in non-commutative variables $X_1, X_2, \ldots, X_n$ with integer coefficients defined as follows. $E(\alpha_i) = 1 + X_i,\ E(\alpha_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \cdots\ (i = 1, 2, \ldots, n)$.

For a sequence $I = i_1 i_2 \ldots i_{k-1} j$ $(i_m \in \{1, 2, \ldots, n\}, k \leq q)$, we define the *Milnor number* $\mu_\gamma(I)$ to be the coefficient of $X_{i_1} X_{i_2} \cdots X_{i_{k-1}}$ in $E(\lambda_j)$ (we define $\mu_\gamma(j) = 0$), which is an invariant [8]. (In [8], the set of λ_j's, without taking the Magnus expansion, is called the *Milnor $\overline{\mu}$-invariant.*) For a bottom tangle $\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$, an oriented ordered link $L(\gamma) = L_1 \cup L_2 \cup \cdots \cup L_n$ in S^3 can be defined by $L_i = \gamma_i \cup a_i$, where a_i is a line segment connecting p_i and q_i, see Figure 2.8.

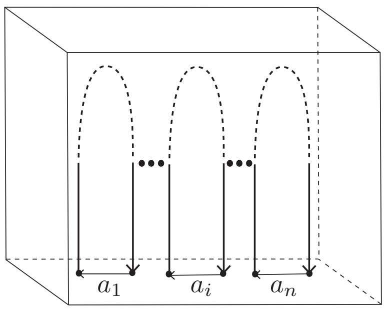


Figure 2.8.

We call $L(\gamma)$ the *closure* of γ. On the other hand, for any link L in S^3, there is a bottom tangle γ_L such that the closure of γ_L is equal to L. So we define the Milnor

number of L to be the Milnor number of γ_L. Let $\Delta_L(I)$ be the greatest common devisor of $\mu_L(J)'s$, where J is obtained from proper subsequence of I by permuting cyclicly. The *Milnor* $\overline{\mu}$*-invariant* $\overline{\mu}_L(I)$ is the residue class of $\mu_L(I)$ modulo $\Delta_L(I)$. We note that for a sequence I, if we have $\Delta_L(I) = 0$, then the Milnor $\overline{\mu}$-invariant $\overline{\mu}_L(I)$ is equal to the Milnor number $\mu_{\gamma_L}(I)$. Now we define the Milnor number for clover links.

Definition 2.1. Let c be an n-clover link and F_c a disk/band surface of c. Let γ_{F_c} be the n-component bottom tangle obtained from F_c. For a sequence I, *the Milnor number* $\mu_c(I)$ *for* c is defined to be the Milnor number $\mu_{\gamma_{F_c}}(I)$.

While $\mu_c(I)$ depends on a choice of F_c, we have the following result.

Theorem 2.2. *Let c be an n-clover link and l_c a link which is the disjoint union of leaves of c. If $\overline{\mu}_{l_c}(J) = 0$ for any sequence J with $|J| \leq k$, then $\mu_c(I)$ is well-defined for any sequence I with $|I| \leq 2k + 1$.*

Remark 2.3. Levine [8] proved the same result as Theorem 2.2 for *flat vertex* clover links under *flatly isotopy.* See Subsection 2.4 for the definition of a flatly isotopy.

2.4. **Proof of Theorem 2.2.** In order to prove Theorem 2.2, we need two Lemmas 2.4 and 2.6.

Two flat vertex graphs Γ and Γ' are *flatly isotopic* if there exists an isotopy $h_t : S^3 \to S^3 (t \in [0, 1])$ such that $h_0 = id$, $h_1(\Gamma) = \Gamma'$ and $h_t(\Gamma)$ is a flat vertex graph for each $t \in [0, 1]$. We call such a isotopy h_t a *flatly isotopy* [15].

An n-component *braid* $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_n$ is a tangle in $[0, 1]^3$ such that for each t $(0 \leq t \leq 1)$, $\displaystyle\bigcup_{i=1}^{n} \beta_i$ intersects $[0, 1] \times [0, 1] \times \{t\}$ transversely at n points. In particular, β is a *pure braid* if for any i $(= 1, 2, \ldots, n)$, the boundary $\partial\beta_i = \{(x, \frac{1}{2}, 0), (x, \frac{1}{2}, 1)\} \subset \partial[0, 1]^3$ for $x \in [0, 1]$.

Lemma 2.4. *If two n-clover links c and c' are ambient isotopic to each other, then there exists an n-clover link c'' such that c'' is obtained from c by a single B-move and c'' is flatly isotopic to c', where a B-move is a local move as illustrated in Figure 2.9.*

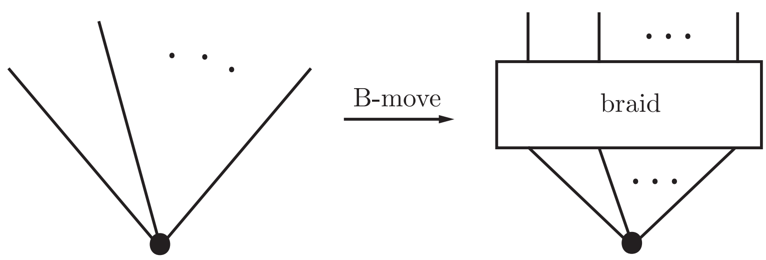


FIGURE 2.9. A B-move

Proof. Consider regular diagrams $\widetilde{c}, \widetilde{c'}$ of c, c' respectively. Since c and c' are ambient isotopic to each other, $\widetilde{c}$ and $\widetilde{c'}$ are related by a finite sequence of Reidemeister moves (i) $\sim$ (v) as illustrated in Figure 2.10 [6].

Moves excepting move (iv) are realized by flatly isotopies. Move (iv) consists of two kinds transformations. One increases a crossing, and the other decreases a

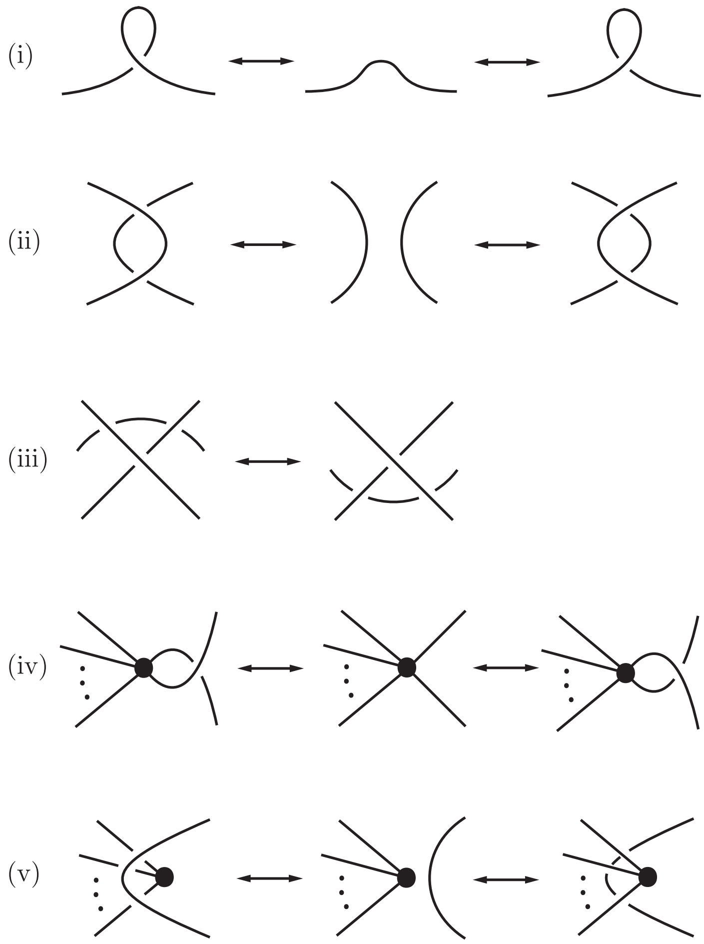


FIGURE 2.10. Reidemeister moves (i)∼(v)

crossing. We call those two kinds of transformations move (iv)$^+$ and move (iv)$^-$ respectively. It is not hard to see that a move (iv)$^-$ is realized by moves (iv)$^+$ and (ii). Hence moves (i) $\sim$ (v) are realized by moves (i) $\sim$ (iii), (v) and (iv)$^+$.

We consider a move (iv)$^+$. If we apply a move (iv)$^+$ to a regular diagram $\widetilde{d}$ of an n-clover link, then we obtain a regular diagram either (a) or (b) as illustrated in Figure 2.11. The diagram (a) is clearly obtained from $\widetilde{d}$ by a single $\widetilde{B}$-move, where $\widetilde{B}$-move is a local move of regular diagrams which correspond to B-move. The diagram (b) is transformed into a diagram (b$'$) as illustrated in Figure 2.12 by a finite sequence of moves (i)∼(iii) and (v). So the diagram (b) is obtained from $\widetilde{d}$ by moves (i)∼(iii), (v) and a $\widetilde{B}$-moves. Thus a move (iv)$^+$ is realized by moves (i)∼(iii), (v) and a $\widetilde{B}$-move. It follows that moves (i) $\sim$ (v) are realized by moves (i) $\sim$ (iii), (v) and $\widetilde{B}$-moves.

Applying a move (v), subsequently a $\widetilde{B}$-move can be realized by applying a $\widetilde{B}$-move, subsequently a move (v) and several moves (ii) and (iii), see Figure 2.13. It is clear that applying each move $(*)$ $(* =$ i, ii, iii$)$, subsequently a $\widetilde{B}$-move can be realized by applying $\widetilde{B}$-moves, subsequently a move $(*)$. So applying several moves (i)∼(iii) and (v), subsequently a $\widetilde{B}$-move can be realized by applying a $\widetilde{B}$-move, subsequently several moves (i)∼(iii) and (v).

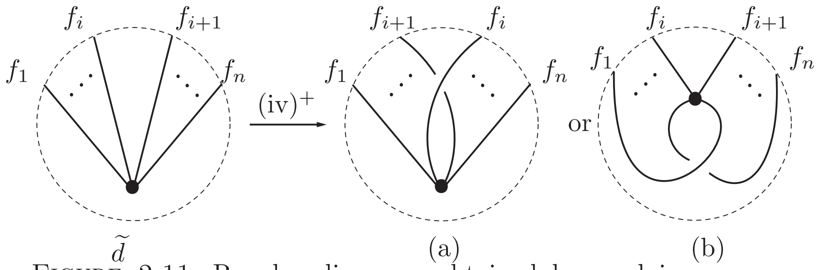


FIGURE 2.11. Regular diagrams obtained by applying a move $(\mathrm{iv})^+$ to a regular diagram $\widetilde{d}$

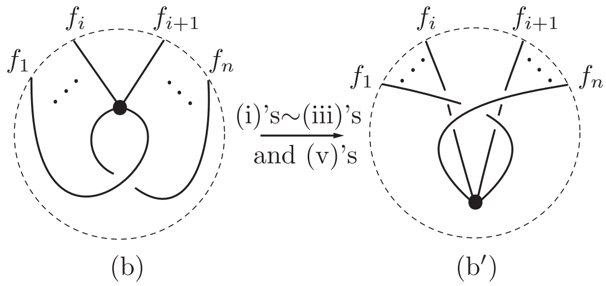


FIGURE 2.12.

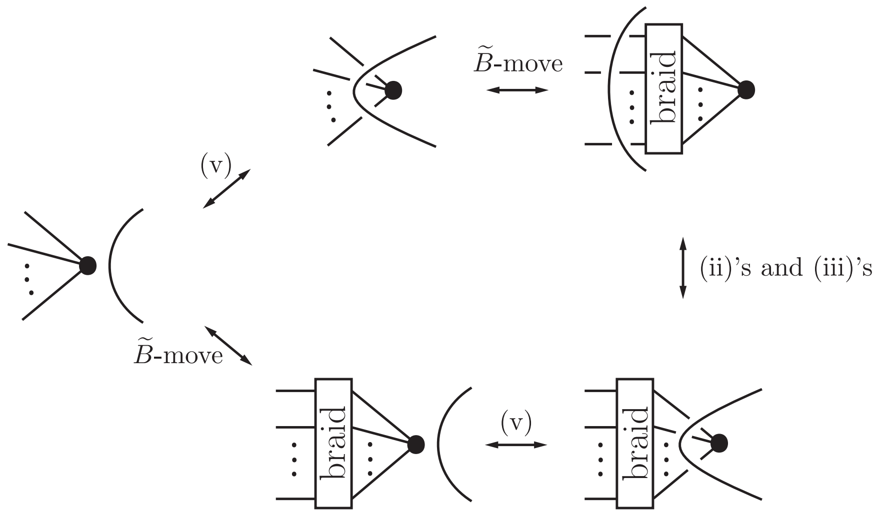


FIGURE 2.13.

Therefore there exists a regular diagram $\widetilde{c''}$ of an n-clover link c'' such that $\widetilde{c''}$ is obtained from $\widetilde{c}$ by a finite sequence of $\widetilde{B}$-moves and $\widetilde{c'}$ is obtained from $\widetilde{c''}$ by a finite sequence of moves (i)∼(iii) and (v). We note that a finite sequence of $\widetilde{B}$-moves is realized by a single $\widetilde{B}$-move. This completes the proof. □

Since we suppose that the disk parts of any disk/band surfaces for a clover link are as illustrated in Figure 2.4, by Lemma 2.4, we immediately have the following Proposition 2.5.

Proposition 2.5. *For an n-clover link c, any two disk/band surfaces F_c and F'_c are transformed into each other by adding full-twists to bands* (*Figure* 2.14 (a)) *and a single move illustrated in Figure* 2.14 (b).

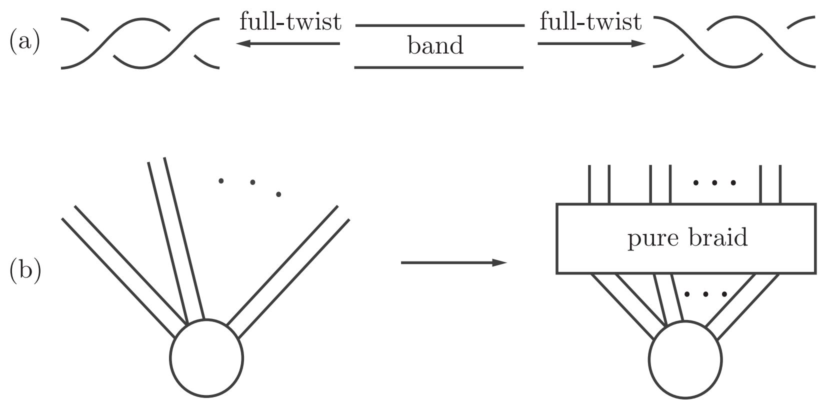


FIGURE 2.14. Two local moves of disk-band surfaces

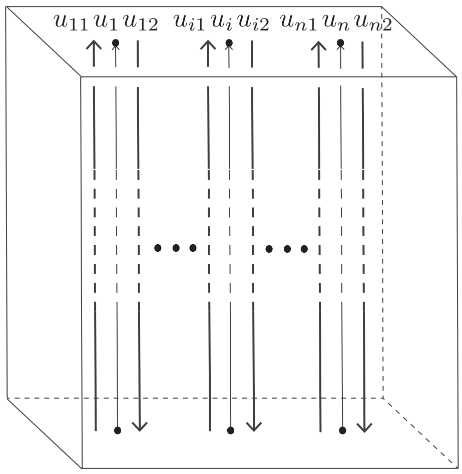


FIGURE 2.15.

Here we introduce a SL-move which is a transformation of an n-component bottom tangle $\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ in $[0,1]^3$

(1) Let $u = u_1 \cup u_2 \cup \cdots \cup u_n$ be an oriented ordered n-component string link in $[0,1]^3$. For each i $(= 1, 2, \ldots, n)$, we consider two arcs u_{i1} and u_{i2} which are parallel to the ith component u_i of u with orientations as illustrated in Figure 2.15. Let $u' = (u_{11} \cup u_{12}) \cup \cdots \cup (u_{n1} \cup u_{n2})$. We may assume that for each i $(= 1, 2, \ldots, n)$, $\partial u_{i1} = \{(\frac{2i-1}{2n+1}, \frac{1}{2}, 0), (\frac{2i-1}{2n+1}, \frac{1}{2}, 1)\}$ and $\partial u_{i2} = \{(\frac{2i}{2n+1}, \frac{1}{2}, 0), (\frac{2i}{2n+1}, \frac{1}{2}, 1)\}$.

(2) Let $\gamma' = \gamma'_1 \cup \gamma'_2 \cup \cdots \cup \gamma'_n$ be an n-component bottom tangle in $[0,1]^3$ defined by
$$\gamma'_i = h_0(u_{i1} \cup u_{i2}) \cup h_1(\gamma_i)$$
for $i = 1, 2, \ldots, n$, where $h_0, h_1 : ([0,1]\times[0,1])\times[0,1] \to ([0,1]\times[0,1])\times[0,1]$ are embeddings defined by

$$h_0(x,t) = (x, \tfrac{1}{2}t) \text{ and } h_1(x,t) = (x, \tfrac{1}{2} + \tfrac{1}{2}t)$$

for $x \in ([0,1] \times [0,1])$ and $t \in [0,1]$.

We say that *γ' is obtained from γ by a SL-move.* We note that if u is trivial, a SL-move is just adding full-twists or nothing. A SL-move is determined by a *String Link* and a number of full-twists; this explains 'SL'in SL-move. For example, see Figure 2.16

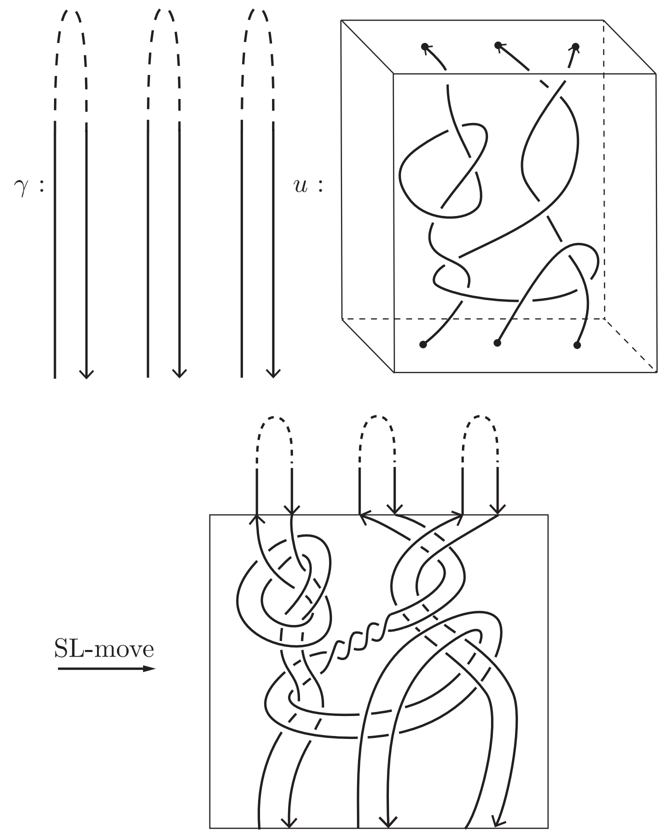


FIGURE 2.16. An example of a SL-move

The following lemma, which is not directly mentioned, can be proved by combining some results in [8]. In the following we directly prove by calculus of Milnor numbers.

Lemma 2.6. [8] *Let γ be an n-component bottom tangle and γ' a bottom tangle obtained from γ by a SL-move. If $\mu_\gamma(J) = \mu_{\gamma'}(J) = 0$ for any sequence J with $|J| \le k$, then $\mu_\gamma(I) = \mu_{\gamma'}(I)$ for any sequence I with $|I| \le 2k+1$.*

Proof. Denote respectively by α_i, λ_i (resp. α'_i, λ'_i) the ith meridian and ith longitude of γ (resp. γ') for $1 \le i \le n$. Let E_X (resp. E_Y) be the Magnus expansion in non-commutative variables $X_1, \ldots, X_n$ (resp. $Y_1, \ldots, Y_n$) obtained by replacing α_j by $1 + X_j$ (resp. α'_j by $1 + Y_j$) and replacing α_j^{-1} by $1 - X_j + X_j^2 - X_j^3 + \cdots$ (resp. α'^{-1}_j by $1 - Y_j + Y_j^2 - Y_j^3 + \cdots$) for $1 \le j \le n$. Let u_i be the ith longitude of a string link which gives the SL-move, see Figure 2.17.

Then we have $\alpha_i = u_i^{-1}\alpha'_i u_i$ and $\lambda'_i = u_i \lambda_i u_i^{-1}$, where α_i and λ_i are assumed to be elements of $\pi_1([0,1]^3 \setminus \gamma')$. Let $\beta_i = [\lambda'_i, \alpha'_i]$ as illustrated in Figure 2.17. By

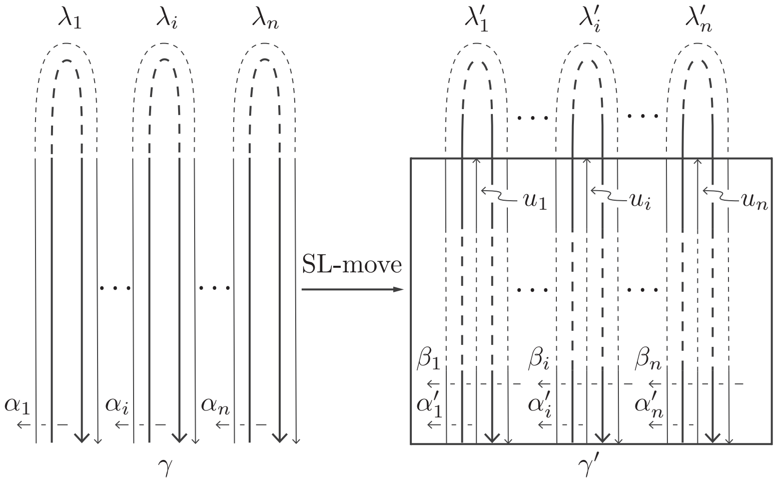


FIGURE 2.17.

the assumption, Milnor numbers for γ and γ' of length $\leq k$ vanish, so $E_X(\lambda_i)$ and $E_Y(\lambda'_i)$ can be written respectively in the form

$$E_X(\lambda_i) = 1 + f_i(X) \text{ and } E_Y(\lambda'_i) = 1 + f'_i(Y),$$

where $f_i(X)$ and $f'_i(Y)$ mean the terms of degree $\geq k$.

First, we observe $E_Y(\alpha_i)$. Let $E_Y(\lambda_i'^{-1}) = 1 + \overline{f'_i}(Y)$, where $\overline{f'_i}(Y)$ is the terms of degree $\geq k$. Note that

$$E_Y(\lambda_i'^{-1})E_Y(\lambda'_i) = (1 + \overline{f'_i}(Y))(1 + f'_i(Y)) = 1.$$

It follows that

$$\begin{array}{rcl} E_Y(\beta_i) & = & E_Y([\lambda'_i, \alpha'_i]) \\ & = & E_Y(\lambda_i'^{-1}\alpha_i'^{-1}\lambda'_i\alpha'_i) \\ & = & (1 + \overline{f'_i}(Y))E_Y(\alpha_i'^{-1})(1 + f'_i(Y))E_Y(\alpha'_i) \\ & = & (1 + \overline{f'_i}(Y))(1 + f'_i(Y) + \mathcal{O}(k+1)) \\ & = & 1 + \mathcal{O}(k+1), \end{array}$$

where $\mathcal{O}(m)$ means the terms of degree $\geq m$. Since u_i is represented by a product of the conjugates of β_i, we have $E_Y(u_i) = 1 + \mathcal{O}(k+1)$ and set

$$E_Y(u_i) = 1 + g_i(Y) \text{ and } E_Y(u_i^{-1}) = 1 + \overline{g_i}(Y),$$

where $g_i(Y)$ and $\overline{g_i}(Y)$ mean the terms of degree $\geq k+1$. As $\alpha_i = u_i^{-1}\alpha'_i u_i$ we have

$$\begin{array}{rcl} E_Y(\alpha_i) & = & E_Y(u_i^{-1}\alpha'_i u_i) \\ & = & (1 + \overline{g_i}(Y))(1 + Y_i)(1 + g_i(Y)) \\ & = & 1 + (1 + \overline{g_i}(Y))Y_i(1 + g_i(Y)) \\ & = & 1 + Y_i + \mathcal{O}(k+2). \end{array}$$

Hence

$$E_Y(\alpha_i) = E_Y(\alpha'_i) + \mathcal{O}(k+2).$$

Now we observe $E_Y(\lambda'_i) - (1 + f_i(Y))$. Since λ_i is represented by a word of $\alpha_1^{\pm1}, \alpha_2^{\pm1}, \ldots, \alpha_n^{\pm1}$, then $E_Y(\lambda_i)$ is obtained from the word by substituting $\alpha_i^{\pm1}$ for $E_Y(\alpha_i^{\pm1}) = E_Y(\alpha_i'^{\pm1}) + \mathcal{O}(k+2)$. Therefore

$$E_Y(\lambda_i) - (1 + f_i(Y)) = \mathcal{O}(k + (k+2) - 1) = \mathcal{O}(2k+1).$$

It follows that

$$\begin{array}{rcl} E_Y(\lambda'_i) & = & E_Y(u_i\lambda_i u_i^{-1}) \\ & = & (1 + g_i(Y))(1 + f_i(Y) + \mathcal{O}(2k+1))(1 + \overline{g_i}(Y)) \\ & = & 1 + f_i(Y) + \mathcal{O}(2k+1). \end{array}$$

This completes the proof. □

Remark 2.7. Let γ, γ' be oriented ordered 4-component bottom tangles illustrated in Figure 2.18. Note that γ' is obtained from γ by a SL-move. By the definition, $\mu_\gamma(j) = \mu_{\gamma'}(j) = 0$ for any sequence j with $|j| = 1(= k)$. By Lemma 2.6, $\mu_\gamma(J) = \mu_{\gamma'}(J)$ for any sequence J with $|J| \leq 3(= 2k+1)$. However, by easy calculus, $\mu_\gamma(1234) = 0 \neq 1 = \mu_{\gamma'}(1234)$ for the sequence 1234 of the length 4(=2k+2). Therefore Lemma 2.6 (Theorem 2.2) generally dose not hold in the case of the length $2k+2$ or more.

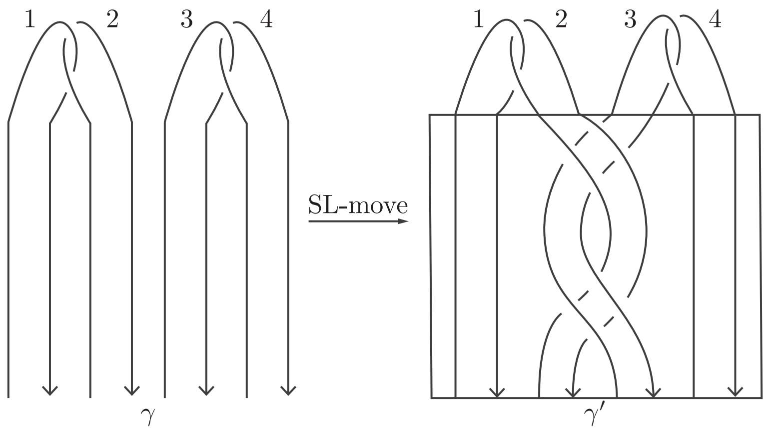


FIGURE 2.18. An example of Lemma 2.6 (Theorem 2.2) dose not hold in the case of the length $2k+2$ or more

By using the above results, we prove Theorem 2.2.

Proof of Theorem 2.2. By Proposition 2.5, any two disk/band surfaces F_c and F'_c of an n-component clover link c are transformed into each other by the moves (a) and (b) illustrated in Figure 2.14. So two bottom tangles γ_{F_c} and $\gamma_{F'_c}$ are transformed into each other by a SL-move. Since the both closures $L(\gamma_{F_c})$ and $L(\gamma_{F'_c})$ are ambient isotopic to l_c and the hypothesis of Lemma 2.6,

$$0 = \overline{\mu}_{l_c}(J) = \mu_{\gamma_{F_c}}(J) = \mu_{\gamma_{F'_c}}(J)$$

for any sequence J with $|J| \leq k$. Hence by Lemma 2.6, $\mu_{\gamma_{F_c}}(I) = \mu_{\gamma_{F'_c}}(I)$ for any sequence I with $|I| \leq 2k+1$. This completes the proof. □

3. Edge-homotopy for clover links

3.1. **Claspers.** Let us briefly recall from [4] the basic notations of clasper theory. In this paper, we essentially only need the notation of C_k-tree. For a general definition of claspers, we refer the reader to [4].

Definition 3.1. Let L be a (clover) link in S^3 (resp. a tangle in $[0,1]^3$). An embedded disk T in S^3 (resp. $[0,1]^3$) is called a *tree clasper* for L if it satisfies the following (1), (2) and (3):
(1) T is decomposed into disks and bands, called *edges*, each of which connects two distinct disks.
(2) The disks have either 1 or 3 incident edges, called *disk-leaves* or *nodes* respectively.
(3) L intersects T transversely and the intersections are contained in the union of the interior of the disk-leaves.
The *degree* of a tree clasper is the number of the disk-leaves minus 1. (In [4], a tree

clasper is called a *strict tree clasper*.) A degree k tree clasper is called a C_k*-tree*. A C_k-tree is *simple* if each disk-leaf intersects L at one point.

We will make use of the drawing convention for claspers of [4, Fig. 4]. Given a C_k-tree T for L, there is a procedure to construct a framed link $\gamma(T)$ in a regular neighborhood of T. *Surgery along* T means surgery along $\gamma(T)$. Since surgery along $\gamma(T)$ preserves the ambient space, surgery along the C_k-tree T can be regarded as a local move on L in S^3 (resp. $[0,1]^3$). We say that the *resulting one* L_T *in* S^3 (*resp.* $[0,1]^3$) *is obtained from* L *by surgery along* T. Similarly, for a disjoint union of trees $T_1 \cup \cdots \cup T_m$, we can define $L_{T_1\cup\cdots\cup T_m}$. Two (string) links (resp. a clover link) are C_k*-equivalence* if they are transformed into each other by surgery along C_k-trees.

The relation between C_k-equivalence and Milnor invariants is known as follows.

Theorem 3.2. [4, Theorem 7.2] *Milnor invariants of length* $\leq k$ *for* (*string*) *links are invariants of* C_k*-equivalence.*

A C_k-tree T for a (string) link (resp. a clover link) L is a *self* C_k*-tree* if all disk-leaves of T intersect the same component (resp. the same spatial edge) of L. The *self* C_k*-equivalence* is an equivalence relation generated by surgery along self C_k-trees. We remark that self C_1-equivalence for a (string) link (resp. a clover link) is the same relation as link-homotopy (resp. edge-homotopy).

3.2. **Edge-homotopy for clover links.** The (edge-homotopy+C_k)-equivalence is an equivalence relation obtained by combining edge-homotopy and C_k-equivalence.

Theorem 3.3. *Let* c, c' *be two* n*-clover links and* $l_c, l_{c'}$ *links which are disjoint unions of leaves of* c, c' *respectively. Suppose that* $\overline{\mu}_{l_c}(J) = \overline{\mu}_{l_{c'}}(J) = 0$ *for any sequence* J *with* $|J| \leq k$*. Then* c *and* c' *are* (*edge-homotopy+*C_{2k+1})*-equivalence if and only if* $\mu_c(I) = \mu_{c'}(I)$ *for any non-repeated sequence* I *with* $|I| \leq 2k+1$*.*

Let c be a clover link and l_c a link which is the disjoint union of leaves of c. Since the union of stems of c and the root is contractible, we may assume that a C_k-tree T for c satisfies $T \cap c = T \cap l_c$. By the zip construction [4] for T, T becomes a disjoint union of simple C_k-trees for l_c. Combining this and [2, Lemma 1.2], for $n \leq m$, if two n-clover links are C_m-equivalence, then they are self C_1-equivalence. Hence by Theorem 3.3, we have the following corollary.

Corollary 3.4. *Let* c, c' *be two* n*-clover links and* $l_c, l_{c'}$ *links which are disjoint unions of leaves of* c, c' *respectively. Suppose that* $\overline{\mu}_{l_c}(J) = \overline{\mu}_{l_{c'}}(J) = 0$ *for any sequence* J *with* $|J| \leq n/2$*. Then* c *and* c' *are edge-homotopic if and only if* $\mu_c(I) = \mu_{c'}(I)$ *for any non-repeated sequence* I *with* $|I| \leq n$*.*

By the definition, the Milnor $\overline{\mu}$-invariant of length 1 is always zero. If $n = 3$, then Corollary 3.4 is not necessary the condition.

Corollary 3.5. *Two 3-clover links* c *and* c' *are edge-homotopic if and only if* $\mu_c(I) = \mu_{c'}(I)$ *for any non-repeated sequence* I *with* $|I| \leq 3$*.*

Example 3.6. Let c and c' are two n-clover links as described in Figure 3.1. Let $l_c, l_{c'}$ be links that are disjoint unions of leaves of c, c' respectively. It is clear that l_c and $l_{c'}$ are ambient isotopic. However, c and c' are not edge-homotopic by Corollary 3.4 because $\mu_c(123) = 1 \neq 0 = \mu_{c'}(123)$.

4. Proof of Theorem 3.3

In order to prove Theorem 3.3, we need the following lemma and a theorem given by [16].

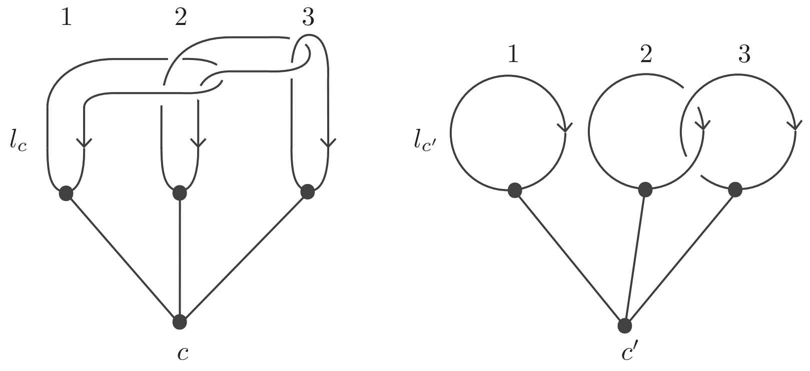


FIGURE 3.1. An example of two clover links which are not edge-homotopic to each other

Lemma 4.1. *Let c, c' be two clover links. Then c and c' are (edge-homotopy+C_k)-equivalence if and only if there exist disk/band surfaces $F_c, F_{c'}$ of c, c' respectively such that the bottom tangles γ_{F_c} and $\gamma_{F_{c'}}$ are (link-homotopy+C_k)-equivalence.*

Proof. We first prove the sufficient condition. By the assumption, there exists a disjoint union G of self C_1-trees and C_k-trees for c such that $c' = c_G$. For c, c', let $F_c, F_{c'}$ be disk/band surfaces of c, c' respectively. We may assume that $F_c \cap G$ is contained in the interior of G. Then we have $\partial F_{c'} = (\partial F_c)_G$ if necessary adding full-twists to bands of F_c. Therefore $\gamma_{F_{c'}} = (\gamma_{F_c})_G$ and G consists of self C_1-trees and C_k-trees for γ_{F_c}.

We next prove the necessary condition. Let S be the union of stems and the root of c, and let $N(S)$ be the regular neighborhood of S. Then we may assume that S is equal to the union of stems and the root of c' and $c \setminus N(S) = \gamma_{F_c}$. Similarly $c' \setminus N(S) = \gamma_{F_{c'}}$. By the assumption, γ_{F_c} and $\gamma_{F_{c'}}$ are (link-homotopy+C_k)-equivalence. This completes the proof. $\square$

Let $\pi : \{1, \dots, k\} \to \{1, \dots, n\} (k \leq n)$ be an injection such that $\pi(1) < \pi(i) < \pi(k) (i \in \{2, \dots, k-1\})$, and $\mathcal{F}_k$ be the set of such injections. For $\pi \in \mathcal{F}_k$, let T_π and $\overline{T}_\pi$ be simple C_{k-1}-trees as illustrated in Figure 4.1, and set V_π (resp. V_π^{-1}) be a string link obtained from the n-component trivial string link by surgery along T_π (resp. $\overline{T}_\pi$). Here, Figure 4.1 are the images of homeomorphisms from the neighborhood of T_π and $\overline{T}_\pi$ to B^3.

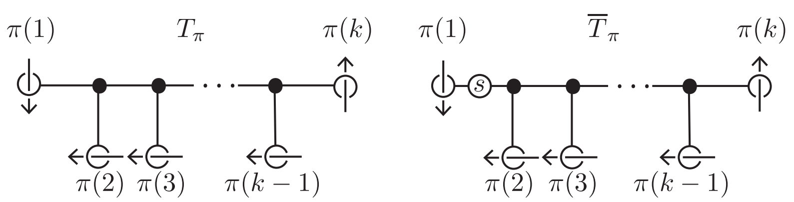


FIGURE 4.1.

Theorem 4.2. [16, Theorem 4.3]. *Let sl be an n-component string link. Then sl is link-homotopic to $sl_1 * sl_2 * \cdots * sl_{n-1}$, where*

$$\begin{aligned} sl_i &= \prod_{\pi\in\mathcal{F}_{i+1}} V_\pi^{x_\pi}, \\ x_\pi &= \begin{cases} \mu_{sl}(\pi(1)\pi(2)) & (i=1), \\ \mu_{sl}(\pi(1)\dots\pi(i+1)) - \mu_{sl_1*\cdots*sl_{i-1}}(\pi(1)\dots\pi(i+1)) & (i\geq 2). \end{cases} \end{aligned}$$

By using the results above, we prove Theorem 3.3.

Proof of Theorem 3.3. We first prove the sufficient condition. Since c and c' are (edge-homotopy+C_{2k+1})-equivalence, by Lemma 4.1, there exist disk/band surfaces $F_c, F_{c'}$ of c, c' respectively such that the n-component bottom tangles γ_{F_c} and $\gamma_{F_{c'}}$ are (link-homotopy+C_{2k+1})-equivalence. Since the Milnor number with a non-repeated sequence is an invariant of link-homotopy [3], by Theorem 3.2, $\mu_c(I) = \mu_{c'}(I)$ with a non-repeated sequence I with $|I| \leq 2k+1$.

Let us prove the necessary condition. Let $F_c, F_{c'}$ be disk/band surfaces of c, c' respectively and let be the $\gamma_{F_c}, \gamma_{F_{c'}}$ be n-component bottom tangles. By Theorem 4.2, two string links which correspond to γ_{F_c} and $\gamma_{F_{c'}}$ are link-homotopic to $sl_1 * sl_2 * \cdots * sl_{n-1}$ and $sl'_1 * sl'_2 * \cdots * sl'_{n-1}$ respectively. Both sl_i and sl'_i are C_i-equivalent to an n-component trivial string link. Set $m = \min\{n, 2k+1\}$, by Theorem 4.2, the two string links which correspond to γ_{F_c} and $\gamma_{F_{c'}}$ are (link-homotopy+C_{2k+1}) to $sl_1 * sl_2 * \cdots * sl_{m-1}$ and $sl'_1 * sl'_2 * \cdots * sl'_{m-1}$ respectively. We recall that if $n \leq 2k+1$, C_{2k+1}-equivalence implies link-homotopy. By the assumption, since $\mu_{\gamma_{F_c}}(I) = \mu_{\gamma_{F_{c'}}}(I)$ for any non-repeated sequence I with $|I| \leq 2k+1$, we have $sl_i = sl'_i$ for each i $(\leq m-1)$. This implies γ_{F_c} and $\gamma_{F_{c'}}$ are (link-homotopy+C_{2k+1})-equivalence. Lemma 4.1 completes the proof. $\square$

References

[1] K. T. Chen, *Commutator calculus and link invariants*, Proc. Amer. Math. Soc. 3, (1952). 44–55.

[2] T. Fleming; A. Yasuhara, *Milnor's invariants and self Ck-equivalence*, Proc. Amer. Math. Soc. 137 (2009), 761–770.

[3] N. Habegger; X. S. Lin, *The classification of links up to link-homotopy*, J. Amer. Math. Soc. 3 (1990), 389–419.

[4] K. Habiro, *Claspers and finite type invariants of links*, Geom. Topol. 4 (2000), 1–83.

[5] K. Habiro, *Bottom tangles and universal invariants*, Algebr. Geom. Topol. 6 (2006), 1113–1214.

[6] L. Kauffman; *Invariants of graphs in three-space*, Trans. Amer. Math. Soc. 311 (1989), 697–710.

[7] L. Kauffman; J. Simon; K. Wolcott; P. Zhao, *Invariants of theta-curves and other graphs in 3-space*, Topology Appl. 49 (1993), 193–216.

[8] J. P. Levine, *The $\overline{\mu}$-invariants of based links*, Differential topology (Siegen, 1987), 87–103, Lecture Notes in Math., 1350, Springer, Berlin, (1988).

[9] J. P. Levine, *An approach to homotopy classification of links*, Trans. Amer. Math. Soc. 306 (1988), 361–387.

[10] V. Magnus; A. Karras; D. Solitar, *Combinatorial Group Theory*, Wiley, New York, London, Sydney, 1966.

[11] J. Milnor, *Link groups*, Ann. of Math. (2) 59, (1954). 177–195.

[12] J. Milnor, *Isotopy of links. Algebraic geometry and topology*, A symposium in honor of S. Lefschetz, pp. 280–306. Princeton University Press, Princeton, N. J., (1957).

[13] J. Stallings, *Homology and central series of groups*, J. Algebra 2 (1965), 170–181.

[14] K. Taniyama, *Cobordism, homotopy and homology of graphs in* $\mathbf{R}^3$, Topology 33 (1994), 509–523.

[15] S. Yamada, *An invariant of spatial graphs*, J. Graph Theory 13 (1989), 537–551.

[16] A. Yasuhara, *Self delta-equivalence for links whose Milnor's isotopy invariants vanish*, Trans. Amer. Math. Soc. 361 (2009), 4721–4749.

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